PhD course on *Partial Differential Equations on Multiple Scales*

I. de Bonis, M. Gahn, P. Henning, M. Neuss Radu, A. Muntean

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Aim of the course. Target audience

The expected audience to this course on *Partial Differential Equations on Multiple Scales* (or *Multiscale PDEs in action*) are PhD students, advanced master students and researchers interested in aspects of multiscale modeling, mathematical analysis and simultation of partial differential equations. The chosen course topics and techniques are meant to prepare the researcher to work on mathematical models based on evolution equations as they arise in material and life sciences. It is advisable that the true beginner looks inside of some of the indicated reading materials (e.g. Refs. [1] or [3] in Gahn's course, Ref. [4] in Henning's course and Ref. [5] in de Bonis 's course.)

Each course will indicate take home assignments. A participation certificate as well as 3 ECTS can be awarded, should this be requested. For this matter, contact Adrian Muntean, prof. dr. habil. (email: adrian.muntean@kau.se).

Course content

The length of the course is cca. 15 hours. There will be three self-content blocks involving different aspects of the mathematical analysis and numerical analysis of (toy) partial differential equations which exhibit either multiscale structures or singularities in production terms. The precise course content and a few bibliographic hints are indicated in the following.

Mini-course on:

Multi-scale analysis for perforated domains including thin heterogeneous layers

Markus Gahn[1]

Heterogeneous media including thin structures play an important role in many applications, e. g. biosciences, medical sciences, geosciences, and material sciences. In such kind of problems different scales occur, which make numerical simulations very expensive or even not possible. Starting from a microscopic model for a problem, the aim is the derivation of macroscopic, or effective, models which give us an approximation of the microscopic problem.

In this course, we deal with mathematical methods from the multi-scale theory for periodically perforated domains which can include thin heterogeneous layers. The basic idea is the postulation of an asymptotic expansion of the microscopic solution [2]. For the rigorous justification of this expansion in a suitable sense, we make use of the two-scale theory which was developed in [1, 4] for domains, and later extended to thin heterogeneous layers in [3]. In the first part of this course, we focus on the principle idea of the asymptotic expansion, and introduce the method of two-scale convergence in domains. We consider the standard two-scale compactness results with simple applications to elliptic problems with oscillating coefficients or in perforated domains. In the second part of the course, we deal with perforated domains including periodic heterogeneous thin layers. Therefore, we extend the method of two-scale convergence in domains to thin heterogeneous structures. As an application, we consider a reaction-diffusion problem in a domain consisting of two bulk domains which are separated by a thin heterogeneous layer with thickness of order ϵ. For $\epsilon \to 0$ the thin layer reduces to a lower dimensional interface Σ. Our aim is the derivation of macroscopic models for $\epsilon \to 0$ with effective interface conditions across Σ.

Key-words: Homogenization; porous Media; two-scale convergence in domains and thin heterogeneous layers; asymptotic expansion; multi-scale analysis.

[1]Hasselt University, Faculty of Sciences - Computational Mathematics, Campus Diepenbeek, Agoralaan Gebouw D BE 3590 Diepenbeek, Belgium.
Email: markus.gahn@uhasselt.be

References

[1] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23:1482–1518, 1992.

[2] A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, 1978.

[3] M. Neuss-Radu and W. Jäger. Effective transmission conditions for reaction-diffusion processes in domains separated by an interface. *SIAM J. Math. Anal.*, 39:687–720, 2007.

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Mini-course on:

Numerical homogenization theory

Patrick Henning[1]

In this mini-course, we will explore elliptic multiscale problems from the perspective of numerical analysis, where we will give special attention to realistic assumptions. In many applications, the considered multiscale coefficients are unstructured (highly heterogenous) and discontinuous, so that the unknown exact solution admits only very low regularity. In order to approach this issue, we will first investigate why conventional discretizations fail to yield reliable approximations if the mesh size does not resolve all variations of the coefficient. In particular we will clarify why coarse finite element spaces, that are theoretically rich enough to contain good L^2-approximations of the unknown multiscale solution, still cannot find such an approximation. Based on the gained insights, we will turn towards "sub-scale corrections" that account for unresolved scales in a given discretization and which allow to replace the original multiscale model by a new "macroscopic" model. This will lead us to the concept of "corrector Green's operators", which can be seen as computable versions of the corrector operators appearing in analytical homogenization theory, but which do not rely on local periodicity or ergodicity. They are based on an orthogonal decomposition (with respect to an energy inner product) of the original solution space

[1]Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden.

(for example $H^1_0(\Omega)$ in the analytical setting or a fine finite element space in the discrete setting) into a coarse space and a detail space. Investigating the decay properties of these "corrector Green's operators", we will be able to localize them to small patches or cells. At the end, we demonstrate how this approach can be turned into a practical numerical method which only requires the solution of a set of (decoupled) patch-problems and one low dimensional (macroscopic) problem at the end. Furthermore, we will show how the information from the corrector operators can be used to turn the obtained L^2-approximation into an accurate H^1-approximation of the original multiscale problem.

Key-words: Numerical homogenization, computational corrector operators, finite elements, LOD methods.

References

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Mini-course on:

Singular elliptic and parabolic problems: existence and regularity of solutions

Ida de Bonis[1]

[1]University Giustino Fortunato, Benevento, Italy.
Email: i.debonis@unifortunato.eu

In this mini-course we will deal with boundary value problems which present lower order terms singular in the variable u that represents the solution.
These terms are singular on the set where $u = 0$ and homogeneous Dirichlet boundary conditions are imposed.
This kind of problems frequently appear in a number of applications. In particular, problems which exhibit lower order terms of type $f(x, u)$ where the nonlinearity is defined for $u \in (0, +\infty)$ while it blows up to infinity when u goes to zero, appear in situations from chemical heterogeneous catalysts, in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, as well as in the theory of heat conduction of electrically conducting materials.
Many authors have been investigating singular problems both stationary and evolution ones. More recently, alike problems has been considered in the weaker framework of Sobolev spaces by L. Boccardo, L. Orsina, D. Giachetti, P. Martinez Aparicio, F. Murat in [1], [2], [3] (see also the references quoted in all these papers).
In this mini-course, we will focus our attention on the existence and the regularity of solutions to some nonlinear parabolic problems which present in the left hand side of the equation a general operator of $p-$laplacian type and in the right hand side of the equation a singular term $F(x, t, u)$ which grows near $u = 0$ as $\frac{1}{u^\gamma}$, $\gamma > 0$.
Problems of this type will be treated with different tools depending on the assumptions on the parameter p, on the sign of the initial datum and also on the fact that the nonlinear term F does have monotonicity properties in s or not. We will also deal with problems with a more general structure in the singular lower order term with respect to the previous case. In particular, such problems exhibit a singular term $F(x, t, u, \nabla u)$ which grows close to $u = 0$ as $\frac{f+D|\nabla u|^q}{u^\gamma}$, $D, \gamma > 0, 1 \leq q \leq p$.
All the results presented can be found in [4] and [5].

Key-words: Weak convergence methods, parameter-independent integral estimates.

References

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[5] I. de Bonis, D. Giachetti, *Nonnegative solutions for a class of singular parabolic problems involving p-laplacian*, Asymptotic Analysis **91** (2015), no. 2, 147-183.