Example-generating tasks in a computer-aided assessment system: Redesign based on student responses

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Based on the patterns of response to an example-generating task, this paper provides suggestions on how the task could be redesigned to enrich students' example spaces in relation to the key ideas addressed by the task. The participants are 236 first-year engineering students.

Keywords: Mathematics education, example-generating tasks, computer-aided assessment.

The past decades have seen rapid development of technologies for automated assessment of students' work in digital environments. In this paper, we use the notion of computer-aided assessment (CAA) systems for this type of technology. Today, CAA systems are in widespread use, particularly in university mathematics courses (Kinnear et al., 2020). However, researchers point out the importance of designing CAA tasks that address higher-order skills in mathematics to prevent assessment solely focusing on lower-order skills (Rønning, 2017). One way to tackle this is to design tasks using the pedagogical approach of prompting students to generate examples that fulfil certain conditions (Kinnear et al., 2020; Yerushalmy et al., 2017). The idea of asking students to generate their own examples has been suggested as a way to foster students' conceptual understanding (Watson & Mason, 2005). Since the responses generated by a group of students most often provide a rich space of examples, it is time-consuming for the teacher to assess student responses. However, by implementing example-generating tasks into a CAA system this correction work could be outsourced (Sangwin, 2003).

The focus of this paper is on a specific type of example-generating task consisting of a sequence of prompts in which a list of constraints are added successively. The aim of the paper is to examine patterns of student response to this type of example-generating task. The findings will inform the task redesign to strengthen the mathematical key ideas addressed by the task, in this case, the Factor theorem and vertical scaling of function graphs.

Example spaces

Central in the teaching and learning of mathematics are *examples*, most often used to introduce a concept or a method (Bills et al., 2006). Watson and Mason (2005) suggest asking students to construct examples that fulfil certain conditions as a powerful approach in the teaching of mathematics. They use the construct of example spaces when referring to the collection of examples provided by students at a given occasion. According to Watson and Mason, the richness of an example space indicates students' mathematical understanding. They point out the importance of encouraging students to extend their existing and accessible example spaces by asking for another, and then another example (Watson & Mason, 2005). Moreover, by asking for several examples that differ as much as possible, students are prompted to generate examples beyond familiar and prototypical ones (Watson & Mason, 2005).

Another way to encourage students to enrich their example spaces, Watson and Mason (2005) argue, is by adding constraints to the initial conditions. In many cases, this "…opens up new possibilities for the learners and promotes creativity." (p. 11). Sangwin (2003) propose this type of example-generating task as particularly appropriate in creating high-level CAA tasks.

Method

The study took place at a Swedish university in autumn 2021 and it involves 236 first-year engineering students taking a course in Calculus. As part of the course assignment, the students conducted small group activities designed for a combined use of a CAA system (*Möbius*) and a dynamic mathematics software (*GeoGebra*). In this paper, we focus on one of the tasks consisting of a sequence of prompts providing constraints one at a time, adopted from Sangwin (2003).

The task

This example-generating task (see Figure 1) was individualized, i.e. students received different numerical values of the parameters (a and b).

Below are some possible properties (i) - (iv) of a polynomial $p(x)$.

(i) $p(x)$ is a polynomial of degree three, i.e. $p(x)$ is a cubic function.
(ii) $p(a) = 0$ (iii) $p(b) = 0$ (iv) $p(0) = ab$

a) Give an example of a polynomial $p(x)$ satisfying (i).
b) Give an example of a polynomial $p(x)$ satisfying (i) and (ii).
c) Give an example of a polynomial $p(x)$ satisfying (i), (ii) and (iii).
d) Give an example of a polynomial $p(x)$ satisfying all the properties (i) - (iv).
e) Give an example of a polynomial $p(x)$ satisfying (ii), (iii) and (iv), but not (i).

Figure 1. Example-generating task adopted from Sangwin (2003).

The main key idea addressed in this task is the Factor theorem, i.e. understanding the relationship between zeros and factors of polynomials. By adding constraints in terms of specific zeros for a polynomial function (Prompt b and Prompt c), the intention is to encourage students to use the Factor theorem. Some students might generate an example to Prompt b without reconsidering the Factor theorem. The addition of a further zero (Prompt c) might foster them to realize the usefulness of using the Factor theorem. Moreover, the intention is to draw students' attention to the possibility of vertical scaling by adding a further constraint in terms of a given y-intercept (Prompt d).

Data collection and analysis

The data consists of student responses collected through the CAA system. In the first stage of the data analysis process, each student response was coded. This analysis generated several codes for each prompt; from 6 (Prompt a) to 18 (Prompt d) different codes. Next, the initial codes were organized into categories guided by the key ideas addressed by the task.

Results

To the first prompt (Prompt a), 36 % of the students (85 out of 236) responded in the most simple way, i.e. $p(x) = x^3$. Predominantly (141/236), the students provided a polynomial in standard form including more than one term. Only 8 students responded in factored form.

The answer categories to Prompt b and Prompt c are the same, except for one. The categories are:

Factor theorem explicitly (FTE). The responses in this category are written in factored form, e.g. $p(x) = x^2(x - a)$ or $p(x) = (x - a)^3$ to Prompt b, and $p(x) = x(x - a)(x - b)$ to Prompt c.

Factor theorem implicitly (FTI). Although the responses in this category are written in standard form, the strategy to generate the answers most probably begins by using the Factor theorem. For example, several students responded $p(x) = x^3 - ax^2$ to Prompt b, most probably by extending $p(x) = x^2(x - a)$. Analogically, to Prompt c, many students responded $p(x) = x^3 - (a + b)x^2 + abx$, which is an extended form of $p(x) = x(x - a)(x - b)$.

Without using the Factor theorem (WFT). To Prompt b, the most common response was $p(x) = x^3 - a^3$, which is straightforward to find without using the Factor theorem. In this category of responses there were also some students that responded with $p(x) = x^3 + x^2 + x - (a^3 + a^2 + a)$.

Undefined (U). There were several responses, written in standard form, in which we were unable to discern the strategies used by the students, i.e. whether they have used the Factor theorem or not.

Table 1 shows an overview of the responses provided to Prompt b and Prompt c. The result indicates that several students provided an example to Prompt b without using the Factor theorem, i.e. the key idea addressed by the task. In total, 37% of the students (87 out of 236) provided a response (to Prompt b) indicating that the Factor theorem has been used. When a further condition (zero) was added (Prompt c), the corresponding proportion of students increased to 78% (183 out of 236).

Table 1. Student responses to Prompt b and Prompt c (numbers within brackets indicate correct answers)

Prompt	FTE	FTI	WFT	Undefined	No answer	Total
Prompt b	50 (50)	37 (35)	74 (71)	73 (62)	2	236 (218)
Prompt c	78 (76)	105 (100)	-	49 (36)	4	236 (212)

By adding a further constraint in terms of a given y-intercept, another key idea is addressed by Prompt d, i.e. the possibility of vertical scaling of a graph by multiplying with a constant factor. Only 14% of the students (33 out of 236) provided responses indicating that they have used this strategy. Most of the students, 58% (136 out of 236), responded with the polynomial $p(x) = (x + 1)(x - a)(x - b)$, either in factored form or standard form. In this way, they received the correct y-intercept without having to use vertical scaling. Notably, as many as 44 students gave this answer already to Prompt c. In total, 27% (64 students) did not provide a correct answer.

When asked to provide an example of a polynomial function that fulfil all the conditions except for being of degree three (Prompt e), most of the students, 80% (189 out of 236), responded with the (correct) second degree polynomial $p(x) = (x - a)(x - b)$, predominantly written in standard form.

To summarize, the findings indicate that the added zero in Prompt c resulted in a significant increase of students utilizing the Factor theorem, i.e. the main key idea addressed by the task. In this respect, the task worked properly. However, in relation to vertical scaling (the other key idea), the findings indicate a need for a redesign of the task. We will elaborate on this in the next section.

Discussion

One reason why most of the students did not need to use vertical scaling to generate an example to Prompt d, we argue, is the feature of the added constraint in this Prompt ($p(0) = ab$). A straightforward way to tackle this would be to revise the constraint to $p(0) = kab$, for some suitable value of the constant k. In this way, the predominant student response, including the factor $(x + 1)$, will require a multiplication with the constant k, i.e. vertical scaling. Another possibility would be to ask students to provide more than one example. This requirement will prompt students who respond with $p(x) = (x + 1)(x - a)(x - b)$, as most of the students in this study did, to extend their example space.

The latter suggestion of redesign is a more general design principle to extend students' example space (Watson & Mason, 2005), which has been adopted to CAA systems (Sangwin, 2003; Yerushalmy et al., 2017). Reconsidering how this design principle could affect other prompts, we argue that this request might encourage many of the students who did not utilize the Factor theorem when responding to Prompt b to do so. For example, students who provided the simple polynomial $p(x) = x^3 - a^3$ will need to extend their example space by using another strategy, hopefully utilizing the Factor theorem. We strongly suggest that the design principle to ask for more than one example is useful in relation to the last prompt (Prompt e) since it was straightforward for students to provide the second-degree polynomial $p(x) = (x - a)(x - b)$. As there is only one second-degree polynomial that fulfils the given conditions, the request for another example will encourage students to consider polynomials of (at least) degree four.

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