Analysis of partial differential equations for moisture transport in porous media

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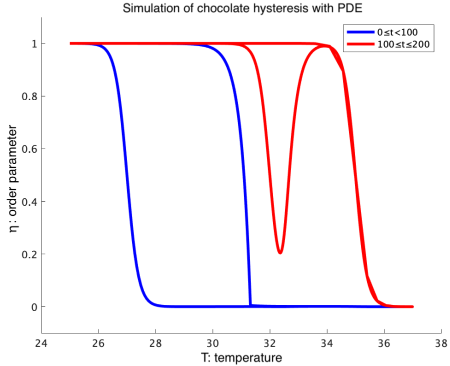
My specialty: Partial Differential Equation

- Chocolate hysteresis (2020-2021).
- Water invasion into porous media (2022-, joint work with EBARA Corporation).

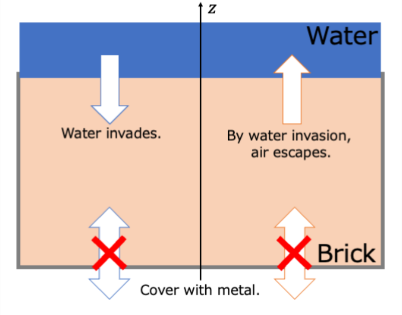


Aim: Analysis for the model describing water adsorption into bricks.

The model is given by Green-Dabiri-Weinaug-Prill(1970).
Assumption:

- Deterioration of bricks in time never occurs.
- Room temperature and humidity do not change.
- All variables are uniformly distributed with respect to the horizontal direction.
  →It is sufficient to consider distribution of water and air on vertical line.

They proposed a system of diffusion equations for water and air in 1D.



For simplicity, we neglect the gravity term.

The GDWP model:

- The unknown functions are the water chemical potential u and the air mass m_a.
- Mass conservation law $\Rightarrow$ two parabolic equations, which describe diffusion with respect to water and air, respectively.

$$\frac{\partial m_w}{\partial t} + \frac{\partial q_w}{\partial z} = 0, \ \frac{\partial m_a}{\partial t} + \frac{\partial q_a}{\partial z} = 0.$$

- The fluxes are obtained from the Darcy law.

$$q_w = -\lambda \left(\frac{\partial P_w}{\partial z} \right), \ q_a = -k_a \left(\frac{\partial P_a}{\partial z} \right).$$

Notation: m_w, m_a: mass, q_w, q_a: flux, λ, k_a: diffusion coefficient, P_w, P_a: pressure.

Note: λ proposed by Fukui et al.(2018) includes exp, it is similar to porous media equation.

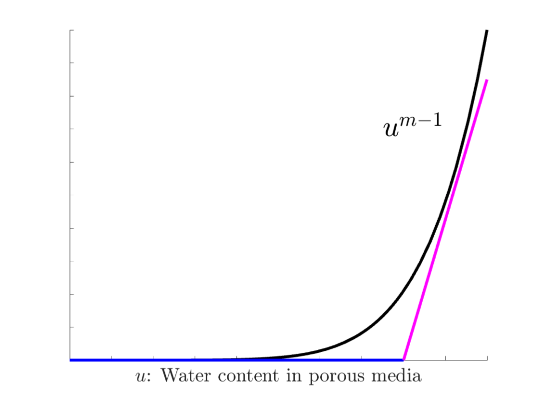
porous medium equation

$$u_t = \Delta(u^m) = \nabla \cdot (m u^{m-1} \nabla u) \ \text{in } \Omega, m > 1.$$

u: water content in porous media, Ω: domain.

【Feature of porous medium equation】

- When the water content is small, diffusion coefficient u^{m-1} is also small.
- When moisture content increases, diffusion coefficient u^{m-1} increases rapidly.



Diffusion coefficient of porous media equation.

λ proposed by Fukui et al.:

$$\lambda = D \cdot \frac{\partial \psi_w}{\partial u}, \; D = D(\psi_w) = \frac{30.332 \times 10^{-6} \exp\left(79.8 \times \psi_w^{1.5}\right)}{\rho_w},$$

$$\psi_w = \psi_w(u) = \frac{0.0505}{8 + \exp\left\{\log_{10}(-u) - 2\right\}} + \frac{0.139}{1.1 + \exp\left\{2.3 \log_{10}(-u) - 4.6\right\}}.$$

$\rightarrow$Is it similar to porous medium equation?

By definition of the water mass $m_w = \rho_w \psi_w$, and the capillary pressure $P_w = \rho_w u + P_a$, we get

$$\rho_w \frac{\partial}{\partial t} \psi_w = \frac{\partial}{\partial x} \left(D(\psi_w) \frac{\partial \psi_w}{\partial u} \frac{\partial}{\partial x} \left(\rho_w u + P_a \right) \right),$$

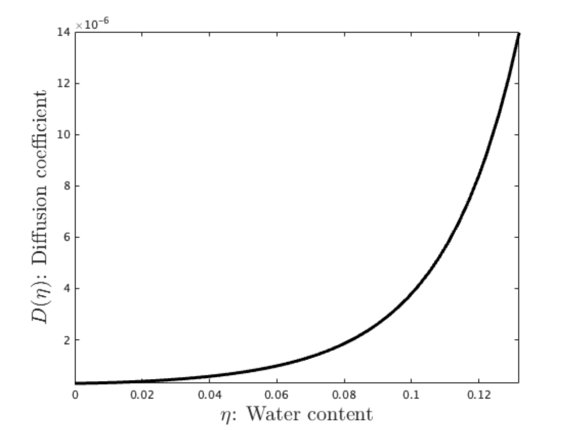
$$\frac{\partial}{\partial t} \psi_w = \frac{\partial}{\partial x} \left(D(\psi_w) \frac{\partial \psi_w}{\partial u} \frac{\partial}{\partial x} \left(u + \frac{P_a}{\rho_w} \right) \right).$$

Notation: ψ_w: water content, ρ_w: water density (constant).

Putting $\eta = \psi_w(u)$, we have

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial x}\left(D(\eta) \left(\frac{\partial \eta}{\partial x} + f(\eta) \right) \right).$$

The graph of $\eta - D(\eta)$ is similar to porous medium equation.

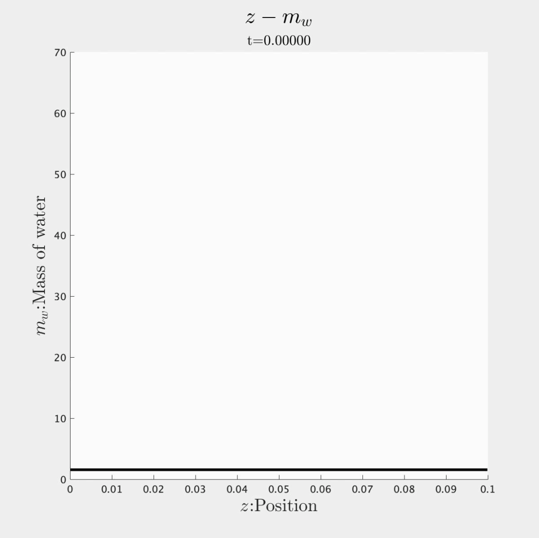


η(water content) - D(η).

numerical calculation(with gravity term)

- We use finite volume method.
- BC of bottom($z = 0$): Flux $= 0$.
  BC of top($z = 0.1$): Flux proportional to pressure difference.
- Initial state is dry brick.

Numerical calculations are done $\rightarrow$ I would like to prove the accuracy of FVM.



Water going into the bricks.

The GDWP model

$$\frac{\partial m_w}{\partial t} + \frac{\partial q_w}{\partial z} = 0, \ \frac{\partial m_a}{\partial t} + \frac{\partial q_a}{\partial z} = 0, \ q_w = -\lambda \left(\frac{\partial P_w}{\partial z} \right), \ q_a = -k_a \left(\frac{\partial P_a}{\partial z} \right),$$

(B.C.), (I.C.).

It is not easy to solve this model directly.

- System of nonlinear parabolic equations.
- Each diffusion coefficients in two parabolic equations depends on u.

$$\lambda = \lambda(u), k_a = k_a(u).$$

- The water pressure and the air pressure depends on two unknown functions.

$$P_w = P_w(u, m_a), P_a = P_a(u, m_a).$$

→We consider only diffusion of water as a first step in this research.

Let $Q(T) = (0, T) \times (0, 1)$.

Initial and boundary value problem (PP)

$$\frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}(u+p)\right) \text{ in } Q(T),$$
$$\frac{\partial}{\partial x}(u+p) = 0 \text{ at } x = 0, 1, \quad u(0, x) = u_0(x) \text{ for } x \in (0, 1).$$

Note: $\psi(u)$ is the water volume fraction, p is the air mass distribution.

- Unknown function $u : Q(T) \to \mathbb{R}$ denotes the water chemical potential
- We assume $\psi, \lambda \in C^2(\mathbb{R})$ and λ is positive and bounded.
- p and u_0 are given functions on $Q(T)$ and $(0, 1)$.

Simplification of the GDWP model:

From the definition of capillary pressure, $P_w = C_w u + P_a$ (C_w is a positive constant). Therefore,

$$q_w = -\lambda\left(\frac{\partial P_w}{\partial x}\right) = -\lambda\left(\frac{\partial}{\partial x}(C_w u + P_a)\right).$$

Since $m_w = C_w \psi(u)$ and by putting $p = \dfrac{P_a}{C_w}$, we get

$$\frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}(u+p)\right).$$

Moreover, $q_w = 0 \text{ at } x = 0, 1$, and $\lambda(u) > 0 \Rightarrow \dfrac{\partial}{\partial x}(u+p) = 0$.

λ in (PP)

$$\frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}(u+p)\right)$$

【Assumption of λ】($\delta_\lambda, C_\lambda$ are constants.)

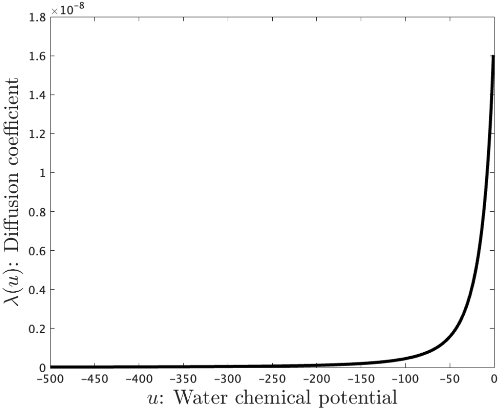
(1) $\lambda \in C^2(\mathbb{R})$,

(2) $\delta_\lambda \le \lambda \le C_\lambda$,

Since λ proposed by Fukui et al. bounded, we assume $\lambda \le C_\lambda$. Also, suppose $\delta_\lambda \le \lambda$ for mathematical reasons.

(3) $|\lambda'|, |\lambda''| \le C_\lambda$,

Clearly, λ and λ' are the Lipschitz continuous.



ψ in (PP)

$$\frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}(u+p)\right)$$

【Assumption of ψ_w (proposed by Fukui et al., not ψ)】 (C_ψ is a constant.)

(1) $\psi_w \in C^2(\mathbb{R})$,

(2) $0 < \psi_w$,

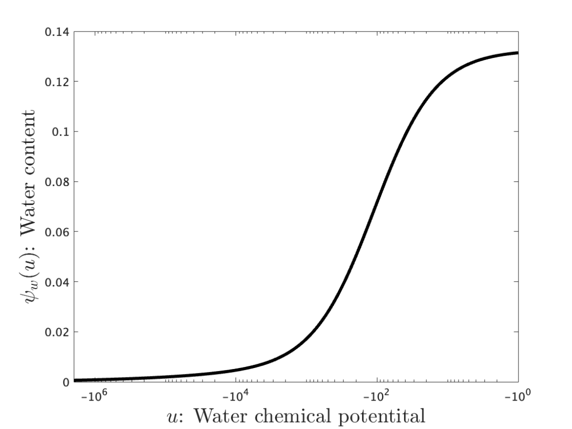
Fitting the water content proposed by Fukui et al.

(3) $0 \leq \psi'_w \leq C_\psi$,

(4) $|\psi''_w| \leq C_\psi$,

ψ'_w is the Lipschitz continuous.

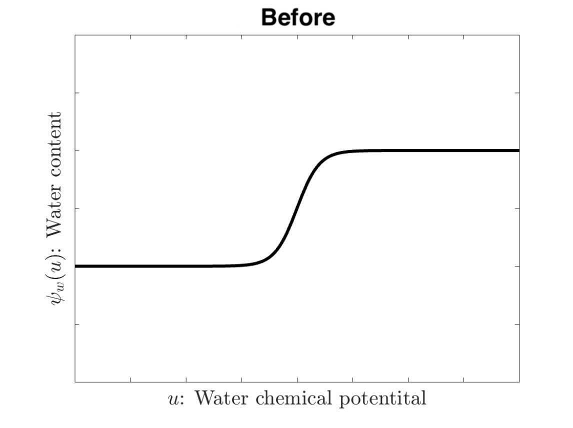
Note: Domain of ψ_w is extended to $\mathbb{R}$.

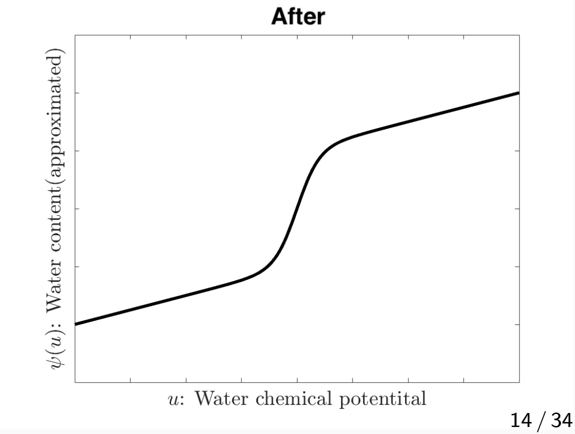


If $u \to \pm\infty$, then $\psi_w' \to 0 \Rightarrow$ Elliptic-parabolic equation (too difficult!)

By approximation $\psi(u) = \psi_w(u) + \varepsilon u$ $(\varepsilon > 0)$, we get $\delta_\psi \leq \psi' \leq C_\psi$ and avoid $\psi_w'(u) \to 0 (u \to \pm\infty)$.

Note: δ_ψ depends on ε.





【Assumption of ψ】

(1) $\psi \in C^2(\mathbb{R})$, (2) $\delta_\psi \le \psi' \le C_\psi$, (3) $|\psi''| \le C_\psi$. (δ_ψ, C_ψ are constants.)

Let $\hat{\lambda}$ be primitive of λ. By putting $v = \hat{\lambda}(u)$, we rewrite (PP) into the following problem, and denote it by (P)(p_x).

Initial and boundary value problem (P)(p_x)

$$\frac{\partial}{\partial t} h(v) = \frac{\partial}{\partial x}(v_x + b(v)p_x) \quad \text{in } Q(T),$$

$$v_x + b(v)p_x = 0 \ \text{ at } x = 0, 1,$$

$$v(0, x) = v_0(x) \ \text{ for } x \in (0, 1),$$

where $h(r) = \psi((\hat{\lambda})^{-1}(r))$, $b(r) = \lambda((\hat{\lambda})^{-1}(r))$ for $r \in \mathbb{R}$ (→diffusion term is linear), $v_0 = \hat{\lambda}(u_0)$.

Remark: Non monotone BC. This shows that it is hard to obtain a strong solution.

$$(\text{BC1}):\ v_x = -b(v)p_x$$

In fact, it is impossible to suppose the sign of $-\dfrac{\partial p}{\partial x}$, since p is the unknown function in the GDWP model. This implies that (BC1) is not monotone. Hence it is hard to obtain strong solutions to auxiliary problems. Namely, it is also hard for (P)(p_x).

In our previous works, we have already obtained a unique strong solution for the following boundary condition (BC2).

$$(\text{BC2}):\ v_x = -b(v)p_x.$$

$$(\mathrm{MP})\begin{cases}\dfrac{\partial}{\partial t}h(v) = \dfrac{\partial}{\partial x}\left(v_x + b(v)p_x\right) \quad \text{in } Q(T),\\ \underbrace{v_x + b(v)p_x = 0}_{\text{(BC2)}} \quad \text{at } x = 0, 1, \quad v(0,x) = v_0(x) \quad \text{for } x \in (0,1).\end{cases}$$

Theorem 1 (Existence and uniqueness of the strong solution)

Let $T > 0, p \in W^{1,2}(0,T; H^2(0,1)), v_0 \in H^1(0,1)$. If (A) holds, then (MP) has a unique solution $v \in W^{1,2}(0,T; L^2(0,1)) \cap L^\infty(0,T; H^1(0,1)) \cap L^2(0,T; H^2(0,1))$.

Assumption (A)

$\psi, \lambda \in C^2(\mathbb{R}), \delta_\psi \le \psi' \le C_\psi, |\psi''| \le C_\psi, \delta_\lambda \le \lambda \le C_\lambda, |\lambda'|, |\lambda''| \le C_\lambda$ on $\mathbb{R}$, where $\delta_\psi, \delta_\lambda, C_\psi, C_\lambda$ are constants.

(Sketch of proof) For $\tilde{u} \in L^2(0,T;H^1(0,1))$:

$$\mathrm{AP}(\tilde{u}) \begin{cases} \frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}u + \lambda(\tilde{u})\frac{\partial}{\partial x}p\right) \text{ in } Q(T), \\ \lambda(u)\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}g = 0 \text{ at } x = 0, 1, \quad u(0,x) = u_0(x) \text{ for } x \in (0,1). \end{cases}$$

For $\mathrm{AP}(\tilde{u})$, we can show the existence of a solution u by applying Kenmochi's theorem (1981). Also, the uniqueness is obtained from the L^1-technique.
We define a mapping $\Gamma_T : L^2(0,T;H^1(0,1)) \to L^2(0,T;H^1(0,1))$ by $\Gamma_T(\tilde{u}) = u$.
Put a set $K(T', M)$ by

$$K(T', M) = \{u \in L^2(0,T;H^1(0,1)); |u_x(t)|^2_{L^2(0,1)} \leq M \text{ a.e. on } (0,T')\}$$

for each $M > 0$, $T' \in (0,T]$.
From some uniform estimates for the solution of $\mathrm{AP}(\tilde{u})$, there exist $M > 0$ and $T' \in (0,T]$ such that $\Gamma_{T'} : K(T', M) \to K(T', M)$.

By the Banach fixed point theorem, (MP) has a unique solution on $[0, T_1]$ for some $T_1 \in (0, T]$. Hence, we can show the existence of the solution on $[0, T]$ by uniform estimates depending only on the initial function.

Furthermore, the uniqueness of the solution is a direct consequence of the Banach fixed point theorem. $\square$

Note : This theorem holds for any positive number T, namely, this theorem means the global existence of the solution in time.

Thus, we get strong solutions for the boundary condition (BC2). However, recently we get a solution for the original boundary condition (BC1) by applying the weak solution.

Definition of a weak solution for (P)(p_x)

If the function v on $Q(T)$ satisfies (C1) and (C2), then v is a weak solution of (P)(p_x).

$$v \in L^\infty(0, T; L^2(0,1)) \cap L^2(0, T; H^1(0,1)). \tag{C1}$$

$$-\int_{Q(T)} h(v)\eta_t dxdt + \int_{Q(T)} (v_x + b(v)p_x)\eta_x dxdr = \int_0^1 h(v_0(0))\eta(0)\, dx, \tag{C2}$$

for $\eta \in W^{1,2}(0, T; L^2(0,1)) \cap L^2(0, T; H^1(0,1))$ with $\eta(T) = 0$.

Theorem 2 (Existence and uniqueness of the weak solution)

If $p \in L^2(0, T; H^1(0,1))$, $v_0 \in L^2(0,1)$ and (A) holds, then (P)(p_x) has at least one weak solution. In addition, suppose $p \in L^4(0, T; H^1(0,1))$, then (P)($p_x$) has at most one weak solution.

(Sketch of proof) Let $p \in L^2(0, T; H^1(0,1))$.
For $\tilde{v} \in L^2(Q(T))$:

$$
\text{(CP1)} \begin{cases} \dfrac{\partial}{\partial t} h(v) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial x} + b(\tilde{v}_\delta)\rho_\delta\right) \text{ in } Q(T), \\ \dfrac{\partial v}{\partial x} + b(\tilde{v}_\delta)\rho_\delta = 0 \text{ at } x = 0, 1, \\ v(0, x) = v_{0\delta}(x) \text{ for } x \in (0, 1). \end{cases}
$$

where $\delta > 0$, $\tilde{v}_\delta = J_\delta * \tilde{v}$, $\rho_\delta = J_\delta * \dfrac{\partial p}{\partial x}$, $v_{0\delta} = J_\delta * v_0$ (J_δ: mollifier, "$*$": convolution).
For (CP1), we can show the existence and uniqueness of a solution v by applying Kenmochi's theorem (1981).

We denote by (CP2) = (CP2)$(\tilde{v})$

$$
\begin{aligned}
&\frac{\partial}{\partial t}h(v) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + b(\tilde{v})\frac{\partial p}{\partial x}\right) \text{ in } Q(T),\\
&\frac{\partial v}{\partial x} + b(\tilde{v})\frac{\partial p}{\partial x} = 0 \text{ at } x = 0, 1,\\
&v(0, x) = v_0(x) \text{ for } x \in (0, 1).
\end{aligned}
$$

From some uniform estimates and the Aubin compact theorem, we can show the existence of a weak solution for (CP2). Also, the uniqueness is obtained by the dual equation method.

Let v be a weak solution of (CP2)$(\tilde{v})$ for $\tilde{v} \in L^2(0, T; L^2(0,1))$. Also, we define a mapping $\Gamma : L^2(0, T; L^2(0,1)) \to L^2(0, T; L^2(0,1))$ by $\Gamma(\tilde{v}) = v$. From some uniform estimates, the Aubin compact theorem and the uniqueness of a solution to (CP2), we see that Γ is continuous. Moreover, let

$$M_k = \{w \in L^2(0, T; L^2(0,1)) | |w|_{L^2(0,T;L^2(0,1))} \leq k\}.$$

Then, there is $\hat{k} > 0$ such that $\Gamma(M_{\hat{k}}) \subset M_{\hat{k}}$. Clearly, $\Gamma(M_{\hat{k}})$ is compact and $M_{\hat{k}}$ is convex.

Therefore, by applying the Schauder fixed point theorem to Γ, the existence of weak solutions to (CP) can be proved. Furthermore, the uniqueness is given by the same method as in (CP2). $\square$
Note: The uniqueness for (CP) requires $p \in L^4(0, T; L^2(0, 1))$.

Since the Schauder fixed point theorem is used, the proof for existence of the solution to (P)(p_x) is not constructive, i.e. existence of a sequence that converges to the fixed point is not guaranteed.
$\rightarrow$ Can we construct a numerical solution that converges to this weak solution?

Finite Volume Method

We consider the following problem:

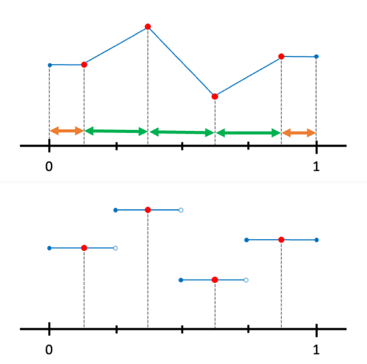
$$L(u) = f \text{ in } \Omega,\ S(u) = 0 \text{ on } \partial\Omega.$$

- Approximate the solution $u(t,x)$ by $\sum_{i=1}^{n} u_i(t)\chi_i(x)$.
- Find u_i from the following integral:

$$\int_\Omega \chi_i(L(u) - f)\,dx = 0 \quad (i = 1, \cdots n).$$

Notation: L: differential operator for u, S: operator regarding BC, Ω: domain, $\partial\Omega$: boundary, n: number of spatial divisions, u_i: value of the numerical solution for each division, χ_i: characteristic function for each division.

Using FVM, n numerical solution values are obtained at a given time for n partitions of the domain.



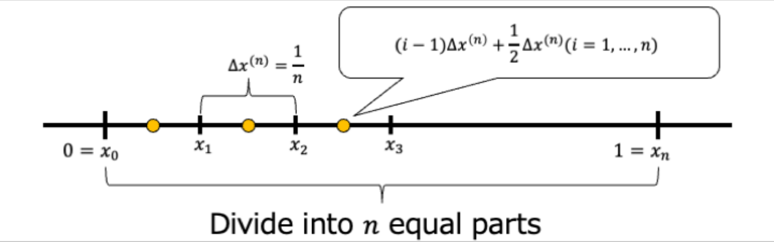
- We adopt a step function not a continuous piecewise linear function as numerical solutions since the number of numerical values is not enough to deal with boundary condition.
- The numerical solution is discontinuous and compact embedding of H^1 into L^2 is not applicable.
- It is easy to handle the boundary condition including flux.

$$+$$

The proof for existence of the solution to (P)(p_x) is not constructive.

$$\Downarrow$$

Does the numerical solution constructed by FVM converge to the weak solution of (P)(p_x)?



χ_i: Characteristic function of $[x_{i-1}, x_i)$

$$v(t,x) \approx \sum_{i=1}^{n} v_i(t)\chi_i(x)$$

By applying FVM to (P)(p_x), we get ordinary differential equation system whose unknown function is tuple of $v_i : [0, T] \to \mathbb{R}$ for $i = 1, \cdots n$.

Integrate both sides of the equation over $x_{i-1} \le x \le x_i$ for $i = 1, \cdots, n$.

$$\underbrace{\int_{(i-1)\Delta x}^{i\Delta x} \frac{\partial}{\partial t} h(v)\, dx}_{I_1} = \underbrace{\int_{(i-1)\Delta x}^{i\Delta x} \frac{\partial}{\partial x} (v_x + b(v)p_x)\, dx}_{I_2},$$

$$I_1 = \int_{(i-1)\Delta x}^{i\Delta x} h'(v) v_t dx \approx \int_{(i-1)\Delta x}^{i\Delta x} h'(v_i) v_{it} dx = h'(v_i) v_{it} \Delta x \quad (1 \le i \le n).$$

- $2 \leq i \leq n-1$

$$I_2 \approx \frac{v_{i+1}(t) - v_i(t)}{\Delta x} + b\left(\frac{v_i(t) + v_{i+1}(t)}{2}\right) p_x(t, i\Delta x)$$
$$- \left(\frac{v_i(t) - v_{i-1}(t)}{\Delta x} + b\left(\frac{v_{i-1}(t) + v_i(t)}{2}\right) p_x(t, (i-1)\Delta x)\right).$$

- $i = 1$

$$I_2 \approx \frac{v_2(t) - v_1(t)}{\Delta x} + b\left(\frac{v_1(t) + v_2(t)}{2}\right) p_x(t, \Delta x).$$

- $i = n$

$$I_2 \approx -\frac{v_n(t) - v_{n-1}(t)}{\Delta x} - b\left(\frac{v_{n-1}(t) + v_n(t)}{2}\right) p_x(t, 1-\Delta x).$$

The initial values are approximated as follows:

$$v_i(0) = \frac{1}{\Delta x}\int_{(i-1)\Delta x}^{i\Delta x} v_0(\xi)\,d\xi \quad (i = 1, \cdots, n).$$

$$
h'(v_i)\frac{\partial v_i}{\partial t} = \begin{cases} \frac{1}{\Delta x}\left(\frac{v_2 - v_1}{\Delta x} + b\left(\frac{v_1 + v_2}{2}\right) p_x(t, \Delta x)\right) & (i = 1), \\ \frac{1}{\Delta x}\left(\frac{v_{i+1} - v_i}{\Delta x} + b\left(\frac{v_i + v_{i+1}}{2}\right) p_x(t, i\Delta x) \right. \\ \qquad \left. - \left(\frac{v_i - v_{i-1}}{\Delta x} + b\left(\frac{v_{i-1} + v_i}{2}\right) p_x(t, (i-1)\Delta x)\right)\right) & (2 \leq i \leq n-1), \\ -\frac{1}{\Delta x}\left(\frac{v_n - v_{n-1}}{\Delta x} + b\left(\frac{v_{n-1} + v_n}{2}\right) p_x(t, (n-1)\Delta x)\right) & (i = n). \end{cases}
$$

Next, we approximate $p_x(t, i\Delta x)(i = 1, \cdots, n-1)$ by

$$
\rho_{\delta,i}^{(n)}(t) = \frac{1}{\Delta x}\int_{(i-1)\Delta x}^{i\Delta x} \rho_\delta(t, x) d\xi \quad (i = 1, \cdots, n-1),
$$

where, $\delta > 0$, $\rho_\delta = J_\delta * p_x$, J_δ: mollifier, "$*$": convolution.

$$h'(v_i)\frac{\partial v_i}{\partial t} = \begin{cases} \frac{1}{\Delta x}\left(\frac{v_2 - v_1}{\Delta x} + b\left(\frac{v_1 + v_2}{2}\right)\rho_{\delta,1}^{(n)}\right) & (i = 1), \\ \frac{1}{\Delta x}\left(\frac{v_{i+1} - v_i}{\Delta x} + b\left(\frac{v_i + v_{i+1}}{2}\right)\rho_{\delta,i}^{(n)}\right. & \\ \qquad \left. - \left(\frac{v_i - v_{i-1}}{\Delta x} + b\left(\frac{v_{i-1} + v_i}{2}\right)\rho_{\delta,i-1}^{(n)}\right)\right) & (2 \le i \le n-1), \\ -\frac{1}{\Delta x}\left(\frac{v_n - v_{n-1}}{\Delta x} + b\left(\frac{v_{n-1} + v_n}{2}\right)\rho_{\delta,n-1}^{(n)}\right) & (i = n), \end{cases}$$

Let (OP) be a problem for the above ordinary differential equation with an initial value approximated by a step function. It is easy to show (OP) has a unique solution. Using the solution $\{v_{\delta,i}^{(n)}\}$ of (OP) and the characteristic function $\chi_i^{(n)}$ of $[x_{i-1}, x_i)$, we define $v_\delta^{(n)}$ as follows:

$$v_\delta^{(n)}(t,x) = \sum_{i=1}^{n} \chi_i^{(n)}(x) v_{\delta,i}^{(n)}(t) \text{ for } (t,x) \in Q(T).$$

Theorem 3 (Construction of numerical solutions)

Let $p \in L^4(0, T; H^1(0,1))$ and (A) holds. For $\delta > 0$, $\{v_\delta^{(n)}\}$ converges to v_δ which is a weak solution of (P)(ρ_δ) as $n \to \infty$. Moreover, $\{v_\delta\}$ converges to v which is a weak solution of (P)(p_x) as $\delta \to 0$.

Lemma (Aubin compact theorem in discrete version) (T. Gallouët, J. -C. Latché, 2011)

Let $z_i^{(n)} \in \mathbb{R}$, $z^{(n)} = \sum_{i=1}^{n} \chi_i^{(n)} z_i^{(n)}$ and

$$\tilde{z}^{(n)}(x) = \begin{cases} 0 & (0 \le x < \Delta x), \\ \dfrac{z^{(n)}(x) - z^{(n)}(x - \Delta x)}{\Delta x} & (\Delta x \le x \le 1). \end{cases}$$

If $\{z^{(n)}\}$ is bounded in $L^\infty(0, T; L^2(0,1))$, $\{\tilde{z}^{(n)}\}$ is bounded in $L^2(0, T; L^2(0,1))$ and $\{z_t^{(n)}\}$ is bounded in $L^2(0, T; H^1(0,1)^*)$, then there exists the subsequence $\{z^{(n_j)}\}$ and $z \in L^2(0, T; L^2(0,1))$ such that $z^{(n_j)} \to z$ in $L^2(0, T; L^2(0,1))$ $(j \to \infty)$.

(Proof of sketch) Our proof is based on uniform estimates and Lemma.

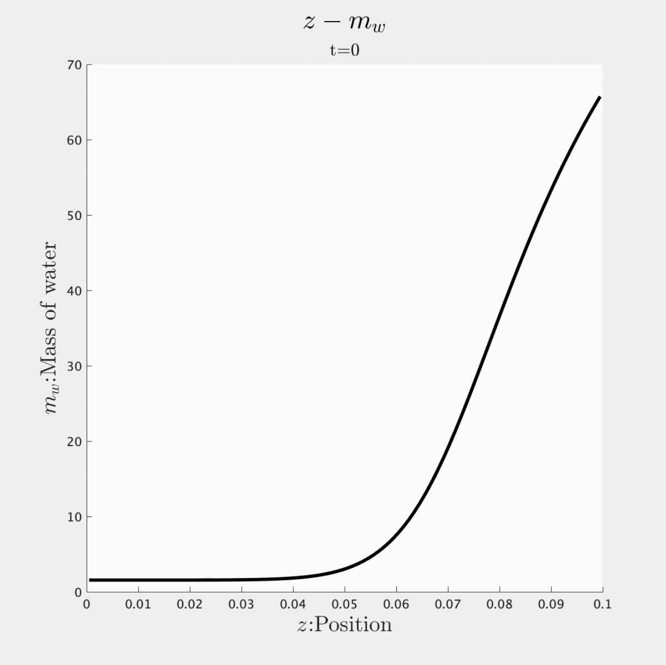
【Numerical calculations to (PP)】

We performed numerical calculations using the specific ψ, λ and so on given in the paper by Fukui et al.

Initial and boundary value problem (PP)

$$\frac{\partial}{\partial t}\psi(u) = \frac{\partial}{\partial x}\left(\lambda(u)\frac{\partial}{\partial x}(u+p)\right) \text{ in } Q(T),$$
$$\frac{\partial}{\partial x}(u+p) = 0 \text{ at } x = 0, 1, \quad u(0,x) = u_0(x) \text{ for } x \in (0,1).$$

- Assume flux $= 0$ both top and bottom.
  $\rightarrow$The initial value of a little water invading into the bricks was given.
- In (PP), we give the function p by solving complete system numerically.


$z - m_w$
t=0
m_w:Mass of water
z:Position
0
10
20
30
40
50
60
70
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Future work

- Error estimate for numerical solutions by FVM.
- Asymptotic behavior of solutions as $\varepsilon \to 0$.
  Note: $\psi(u) = \psi_w(u) + \varepsilon u$.
- Relax the assumption for λ, ψ.
- Existence and uniqueness of solutions of (PP) in the 3D domain.
- Solvability of the system of the partial differential equations with unknown mass distributions in water and air.

Thank you for your attention!